

Some remarks on a system of quasilinear elliptic equations

Lucio BOCCARDO
Università di Roma I
Dipartimento di Matematica
Piazza A. Moro 2
00185 Roma - Italia
e-mail: boccardo@mat.uniroma1.it
Djairo GUEDES DE FIGUEIREDO
IMECC-UNICAMP
Caixa Postal 6065
13081-970 - Campinas-SP
Brasil
e-mail: djairo@ime.unicamp.br

Abstract. We study the existence of critical points (minima and saddle points) of functionals of the type

$$\Phi(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla v|^q - \int_{\Omega} F(x, u, v),$$

where p and q are real numbers larger than 1.

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1 Introduction

In this paper we study the functional

$$\Phi(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla v|^q - \int_{\Omega} F(x, u, v), \quad (1.1)$$

where p and q are real numbers larger than 1, Ω is some bounded domain in R^N , u and v are real-valued functions defined in $\overline{\Omega}$ and belonging to appropriate spaces of functions and F (sometimes referred as a *potential*) is a real-valued differentiable function with domain $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Our aim is to study the geometry of this functional viewing to determining its critical points. Such critical points are the solutions of associated Euler-Lagrange equations, which in the present case is the system of quasilinear elliptic equations below

$$\begin{aligned} -\Delta_p u &= F_u(x, u, v) \\ -\Delta_q v &= F_v(x, u, v) \end{aligned} \quad (1.2)$$

where F_u designates the partial derivative of F with respect to u and Δ_p is the so-called p -Laplacian operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

The geometry of Φ is sort of similar to the one of the functional

$$\frac{1}{p} \int |\nabla u|^p - \int F(x, u),$$

which corresponds to a single quasilinear equation. However, some interesting features appear due to the coupling in the equations (1.2). Our theorems include and unify some previous results by Boccardo – Fleckinger de Thelin [BFT] and de Th  lin-V  lin [VT].

Let us introduce the precise assumptions under which our problem is studied. Our functional Φ is to be defined in the Cartesian product of Sobolev spaces $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. For that matter, the following assumption on F has to be made, although stronger restriction will come timely:

$$\begin{aligned} F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \text{ is } C^1 \text{ and} \\ |F(x, u, v)| &\leq c(1 + |u|^{p^*} + |v|^{q^*}), \end{aligned} \quad (F_1)$$

where $p^* = pN/(N - p)$ and c is some positive constant. Similarly q^* .

In this work, we assume that both p and q are less than N . If this is so, we have the continuous imbeddings $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$, which then tell us that Φ is well defined. In order to have it of class C^1 , we require a stronger condition than (F_1) , namely: F is C^1 and

$$\begin{aligned} |F_u(x, u, v)| &\leq C(1 + |u|^{p^*-1} + |v|^{\frac{q^*(p^*-1)}{p^*}}) \\ |F_v(x, u, v)| &\leq C(1 + |v|^{q^*-1} + |u|^{\frac{p^*(q^*-1)}{q^*}}). \end{aligned} \quad (F_2)$$

It is easy to prove that, if (F_2) is satisfied, then also is (F_1) . Under (F_2) , it follows that the critical points of Φ are the weak solutions of system (1.2), subject to

Dirichlet boundary conditions. For easy reference, we summarize the aforesaid as follows: • under hypothesis (F₂), with $E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, we have

$\Phi : E \rightarrow \mathbb{R}$ is a C^1 -functional.

The geometry of Φ depends strongly on the values of r and s in the estimate below

$$|F(x, u, v)| \leq c(1 + |u|^r + |v|^s), \quad (\text{F}_3)$$

where c is some positive constant, and $r \leq p^*, s \leq q^*$. We discuss three distinct cases

- (I) $r < p$ and $s < q$, (“sublinear-like”)
- (II) $r > p$ or $s > q$, and $r < p^*, s < q^*$, (“superlinear-like”)
- (III) $r = p$ and $s = q$, (“of resonant-type”).

The expressions in parenthesis are to remind us of similar terminology in the case $p = q = 2$. Of course, there are several other situations, which could be of interest to consider. Observe that we are considering only subcritical cases. The cases where either $r = p^*$ or $s = q^*$ or both equalities hold lead to a loss of compactness, and to problems which should be investigated.

Next we state the main results of this paper.

Theorem 1 (The coercive case). *Assume (F₂) and (F₃) with r and s as in (I). Then Φ achieves a global minimum at some $(u_0, v_0) \in E$, which is then a weak solution of system (1.2).*

If we are in the situation that

$$F(x, 0, 0) = F_u(x, 0, 0) = F_v(x, 0, 0), \text{ for all } x \in \overline{\Omega}, \quad (\text{F}_4)$$

then $u \equiv 0$ and $v \equiv 0$ are a trivial solution of system (1.2). In this case the relevant question is obtaining a non-trivial solution of (1.2). This will be possible under appropriate assumptions on the function F , as it is stated in the next results.

Theorem 2 (The coercive case, non-trivial solution). *Assume (F₂), (F₄) and (F₃) with r and s as in (I). Then Φ achieves a global minimum at a point $(u_0, v_0) \neq (0, 0)$, provided there exist positive constant R and $\theta < 1$, and a continuous function $K : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} F(x, t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) &\geq t^\theta K(x, u, v), \text{ for } x \in \overline{\Omega}, |u|, |v| \\ &\leq R, \text{ and small } t > 0. \end{aligned} \quad (\text{F}_5)$$

In this case a Palais-Smale condition holds (see Lemma 4.1) if we assume that there are numbers $R > 0$, θ_p and θ_q with

$$\frac{1}{p^*} < \theta_p < \frac{1}{p} \quad \frac{1}{q^*} < \theta_q < \frac{1}{q}$$

such that

$$0 < F(x, u, v) \leq \theta_p u F_u(x, u, v) + \theta_q v F_v(x, u, v) \quad (\text{F}_6)$$

for all $x \in \bar{\Omega}$ and $|u|, |v| \geq R$.

Theorem 3 Assume (F_2) , (F_4) , (F_6) and (F_3) with r and s as in (II). Assume also that there are constants $c > 0$ and $\varepsilon > 0$ and numbers $\bar{r} > p$, $\bar{s} > q$, such that

$$|F(x, u, v)| \leq c(|u|^{\bar{r}} + |v|^{\bar{s}}), \text{ for } |u|, |v| \leq \varepsilon, x \in \bar{\Omega}. \quad (\text{F}_7)$$

Then Φ has a non-trivial critical point.

Remark. Without loss of generality we may assume $\bar{r} < p^*$ and $\bar{s} < q^*$.

Next we study the situation when (F_3) holds with r and s as in (III), the case we called “of resonant type”. In this case, it is quite adequate to assume a condition on F that implies that the functional Φ satisfies the so-called Cerami condition (see Section 4 for the definition). The assumption on F is: there are positive numbers c, R, μ and ν such that

$$\frac{1}{p} u F_u + \frac{1}{q} v F_v - F \geq c(|u|^\mu + |v|^\nu) \quad (\text{F}_8)$$

for $|u|, |v| > R$.

This type of condition has been introduced by Costa-Magalhães [CM1], [CM2]. In order to avoid resonance we shall assume a condition on F involving an eigenvalue problem, which we introduce next. Let $G : \mathbb{R}^2 \rightarrow [0, \infty)$ be a C^1 even function such that

$$G(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) = t G(u, v) \quad (\text{G}_1)$$

$$G(u, v) \leq k(|u|^p + |v|^q). \quad (\text{G}_2)$$

Examples of such functions are

- (i) $G(u, v) = c_1 |u|^p + c_2 |v|^q$
- (ii) $G(u, v) = c |u|^\beta |v|^\gamma$, with $\frac{\beta}{p} + \frac{\gamma}{q} = 1$,

where c_1, c_2 and c are positive constants.

We shall prove in Section 3 that the eigenvalue problem

$$-\Delta_p u - aG_u = \lambda|u|^{p-2}u$$

$$-\Delta_q v - aG_v = \lambda|v|^{q-2}v$$

subject to Dirichlet boundary conditions, with $a \in L^\infty(\Omega)$, has an eigenvalue $\lambda_1(a)$, characterized variationally by

$$\frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int aG(u, v) \geq \lambda_1(a) \left(\frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q \right)$$

for all $(u, v) \in E$.

Now we introduce the following assumption:

$$\lambda_1(a) > 0, \text{ where } \limsup_{|u|, |v| \rightarrow \infty} \frac{F(x, u, v)}{G(u, v)} \leq a(x) \in L^\infty(\Omega) \quad (\text{F}_9)$$

and state the next results.

Theorem 4 Assume (F_2) , (F_8) , (F_9) and (F_3) with r and s as in (III). Then the functional Φ is bounded below and the infimum is achieved.

Theorem 5 Assume (F_2) , (F_4) , (F_8) and (F_3) with r and s as in (III). Suppose also that there are positive numbers R and ε , and $L^\infty(\Omega)$ functions $b(x)$ and $c(x)$ such that

$$\lambda_1(b) < 0, \quad F(x, u, v) \geq b(x)G(u, v), \quad |u|, |v| \geq R \quad (\text{F}_{10})$$

$$\lambda_1(c) > 0, \quad F(x, u, v) \leq c(x)\tilde{G}(u, v), \quad |u|, |v| \leq \varepsilon, \quad (\text{F}_{11})$$

where G and $\tilde{G}$ are functions satisfying the conditions (G_1) and (G_2) . Then, the functional Φ possesses a non-trivial critical point.

2 Special classes of potentials F

(i) The following class of potentials F (and its perturbations) have been considered by [dT], [VT], [FMT]:

$$F(x, u, v) = c(x)|u|^\beta|v|^\gamma$$

where $c(x) \in L^\infty(\overline{\Omega})$ and $\beta, \gamma \geq 1$. Using Young's inequality

$$|u|^\beta |v|^\gamma \leq \frac{1}{m} |u|^{\beta m} + \frac{1}{n} |v|^{\gamma n}$$

where $1/m + 1/n = 1$. Let $r = \beta m$ and $s = \gamma n$. So $\frac{\beta}{r} + \frac{\gamma}{s} = 1$. Consequently, for this class the three cases are

$$\frac{\beta}{p} + \frac{\gamma}{q} < 1. \quad (\text{I})_i$$

$$\frac{\beta}{p} + \frac{\gamma}{q} > 1 \quad \text{and} \quad \frac{\beta}{p^*} + \frac{\gamma}{q^*} < 1. \quad (\text{II})_i$$

$$\frac{\beta}{p} + \frac{\gamma}{q} = 1. \quad (\text{III})_i$$

In this example, condition (F_5) is precisely the inequality in (I) above. Condition (F_6) holds if β and γ are such that

$$\theta_p \beta + \theta_q \gamma \geq 1. \quad (2.1)$$

Observe that, if (2.1) holds with θ_p and θ_q as in the Introduction, then we are necessarily in case (II). That is, the problem is “superlinear-like”.

The theorems stated in the Introduction contain and extend some of the results of the above mentioned papers.

(ii) In [BFT] the following system was studied

$$\begin{aligned} -\Delta_p u &= a(x)|u|^{\alpha-2}u + b(x)|v|^{\beta-2}v + f \\ -\Delta_q v &= c(x)|u|^{\gamma-2}u + d(x)|v|^{\delta-2}v + g \end{aligned}$$

subject to Dirichlet boundary conditions. In this generality, the system is not variational. However, if $b(x) = c(x)$ and $\beta = \gamma = 2$, the above equations are the Euler Lagrange equations of a functional Φ as in (1.1) with

$$F(x, u, v) = a(x)|u|^\alpha + b(x)uv + d(x)|v|^\delta + fu + gv,$$

where we assume that $a, b, d \in L^\infty(\Omega)$, $\alpha, \delta \geq 1$ and $f \in L^{(p^*)}'(\Omega)$, $g \in L^{(q^*)}'(\Omega)$. Here $(p^*)' = \frac{pN}{p+N(p-1)}$ and a similar expression for $(q^*)'$. The fact that f and g are not necessarily in $L^\infty(\Omega)$ implies that the various pointwise estimates (F) cannot hold. However, since the terms where they appear are linear in u and v , their presence essentially do not change the proofs of the theorems. So, in this example, the three cases studied are:

$$\alpha < p, \delta < q, \frac{1}{p} + \frac{1}{q} < 1. \quad (\text{I})_{ii}$$

$$\alpha > p \text{ or } \delta > q \text{ or } \frac{1}{p} + \frac{1}{q} > 1. \quad (\text{II})_{\text{ii}}$$

$$\alpha = p, \delta = q, \frac{1}{p} + \frac{1}{q} = 1. \quad (\text{III})_{\text{ii}}$$

We remark that case (II)_{ii} was not considered in [BFT]. Our results for case (III)_{ii} extend the ones in [BFT].

Remark. For the special examples above, condition (F5) can be replaced by (F5)' there are positive constants c and ε such that

$$F(x, t^q, t^p) > ct^{pq} \quad \text{for all } x \in \overline{\Omega}, 0 < t < \varepsilon.$$

3 The eigenvalue problem

Let $G : R^2 \rightarrow [0, \infty)$ be a C^1 even function satisfying conditions (G₁) and (G₂) given in the Introduction.

Lemma 3.1 *Given $a \in L^\infty(\Omega)$, there are a real number $\lambda_1(a)$ and $(u_0, v_0) \in E$, such that*

$$\begin{cases} -\Delta_p u_0 - aG_u(u_0, v_0) = \lambda_1(a)u_0|u_0|^{p-2} \\ -\Delta_q v_0 - aG_v(u_0, v_0) = \lambda_1(a)v_0|v_0|^{q-2} \end{cases} \quad (3.1)$$

and

$$\frac{1}{2} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int aG(u, v) \geq \lambda_1(a) \left[\frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q \right] \quad (3.2)$$

for all $(u, v) \in E$, with equality for (u_0, v_0) .

Proof. Choose $M > k\|a\|_{L^\infty}$, where k is the constant in (G₂). Then the functional

$$\begin{aligned} J(u, v) &= \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int aG(u, v) \\ &\quad + M \left[\frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q \right] \end{aligned} \quad (3.3)$$

is non-negative for $(u, v) \in E$. Let

$$S = \left\{ (u, v) \in E : \frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q = 1 \right\} \quad (3.4)$$

and let us look for $\inf\{J(u, v) : (u, v) \in S\}$. Let us denote this infimum by μ , and let us take a minimizing sequence $(u_n, v_n) \in S$. It follows that $\|u_n\|_{W^{1,p}}$ and

$\|v_n\|_{W^{1,q}}$ are bounded. So we may choose subsequences (denoted again by (u_n) and (v_n)) such that (u_n) converges to u_0 , weakly in $W_0^{1,p}$ and strongly in L^p . Similarly for (v_n) . Passing to the limit

$$\begin{aligned} & \frac{1}{p} \int |\nabla u_0|^p + \frac{1}{q} \int |\nabla v_0|^q - \int a + G(u_0, v_0) \\ & + M \left[\frac{1}{p} \int |u_0|^p + \frac{1}{q} \int |v_0|^q \right] \leq \mu \end{aligned}$$

which indeed is an equality because $(u_0, v_0) \in S$. So the above infimum is achieved. It follows then that

$$\begin{cases} -\Delta_p u_0 - aG_u(u_0, v_0) + Mu_0|u_0|^{p-1} = \mu_M u_0|u_0|^{p-1} \\ -\Delta_q v_0 - aG_v(u_0, v_0) + Mv_0|v_0|^{q-1} = \mu_M v_0|v_0|^{q-1} \end{cases} \quad (3.5)$$

where μ_M is the Lagrange multiplier. It follows from (G1) that

$$G(u, v) = \frac{1}{p} uG_u(u, v) + \frac{1}{q} vG_v(u, v) \quad (3.6)$$

for all $(u, v) \in R^2$. From the minimization above we have:

$$\begin{aligned} & \mu \left[\frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q \right] \\ & \leq \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int aG(u, v) \\ & \quad + M \left[\frac{1}{p} \int |u|^p + \frac{1}{q} \int |v|^q \right] \end{aligned} \quad (3.7)$$

for all $(u, v) \in E$. It follows from (3.5), (3.6) and (3.7) that $\mu = \mu_M$. In this way we get (3.1) and (3.2) with $\lambda_1(a) = \mu - M$, and the eigenfunction pair (u_0, v_0) as obtained above. $\square$

Remark. From the minimization argument above, it follows that both u_0 and v_0 can be taken ≥ 0 in Ω . As in [dT] it can be proved that u_0, v_0 are $C^1(\overline{\Omega})$ as consequence of regularity results of [T]. Then by the Vasquez Maximum Principle [V] for the p -Laplacian, we conclude that u_0 and v_0 are indeed > 0 in Ω .

Lemma 3.2 $\lambda_1(a)$ is a continuous function of a in the L^∞ norm.

Proof. Let us denote by J_a the functional defined in (3.3) and J_b the corresponding one with a replaced by b , where b is some other L^∞ -function. Let $\lambda_1(b)$ be eigenvalue corresponding to b given in Lemma 3.1. We show that, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|\lambda_1(a) - \lambda_1(b)| < \varepsilon$ provided $\|a - b\|_{L^\infty} \leq \delta$.

Indeed, given $\varepsilon > 0$, choose $(u_\varepsilon, v_\varepsilon) \in S_1$ such that

$$J_a(u_\varepsilon, v_\varepsilon) \leq \lambda_1(a) + \frac{\varepsilon}{2}. \quad (3.8)$$

Next (G_2) implies

$$G(u, v) \leq K \left(\frac{1}{p} |u|^p + \frac{1}{q} |v|^q \right) \quad (3.9)$$

where $K \geq \max\{kp, kq\}$. Using (3.9) we obtain

$$|J_b(u_\varepsilon, v_\varepsilon) - J_a(u_\varepsilon, v_\varepsilon)| \leq \|b - a\|_{L^\infty} K. \quad (3.10)$$

So, from (3.10) and (3.8)

$$\lambda_1(b) \leq J_b(u_\varepsilon, v_\varepsilon) \leq J_a(u_\varepsilon, v_\varepsilon) + K\|b - a\|_{L^\infty} \leq \lambda_1(a) + \frac{\varepsilon}{2} + K\|b - a\|_{L^\infty}$$

Choosing $\delta = \frac{\varepsilon}{2k}$ we have $\lambda_1(b) \leq \lambda_1(a) + \varepsilon$.

Reversing the roles of a and b , the result follows. $\square$

4 Compactness conditions

We say that $\Phi : E \rightarrow \mathbb{R}$ satisfies the *(PS) condition* if, all $(u_n, v_n) \in E$ such that

$$|\Phi(u_n, v_n)| \leq \text{const} \quad \Phi'(u_n, v_n) \rightarrow 0 \quad (4.1)$$

contains a convergent subsequence in the norm of E .

Lemma 4.1 *Suppose that F satisfies (F_2) , (F_6) and (F_3) as in (II). Then the functional Φ defined in the Introduction satisfies the *(PS) condition*.*

Proof. It follows from (4.1) that

$$\begin{aligned} \left| \frac{1}{p} \int |\nabla u_n|^p + \frac{1}{q} \int |\nabla v_n|^q - \int F(u_n, v_n) \right| &\leq \text{const}, \\ \left| \int |\nabla u_n|^p - \int F_u(u_n, v_n) u_n \right| &\leq \varepsilon_n \|u_n\|_{W^{1,p}} \end{aligned}$$

and a similar one for v_n , where $\varepsilon_n \rightarrow 0$. Using these expressions we get

$$\begin{aligned} &\left(\frac{1}{p} - \theta_p \right) \int |\nabla u_n|^p + \left(\frac{1}{q} - \theta_q \right) \int |\nabla v_n|^q - \int (F(u_n, v_n) - \theta_p u_n F_u - \theta_q v_n F_v) \\ &\leq c + c(\|u_n\|_{W^{1,p}} + \|v_n\|_{W^{1,q}}) \end{aligned}$$

which implies, using (F₆) that both $\|u_n\|_{W^{1,p}}$ and $\|v_n\|_{W^{1,q}}$ are bounded. The existence of convergent subsequences follows in a standard way, since the growth of F is below the critical exponents p^* and q^* . $\square$

We say that $\Phi : E \rightarrow \mathbb{R}$ satisfies the *Cerami condition* (Ce) for short, if all $(u_n, v_n) \in E$ such that

$$|\Phi(u_n, v_n)| \leq \text{const} \quad (1 + \|u_n\|_{W^{1,p}} + \|v_n\|_{W^{1,q}})\Phi'(u_n, v_n) \rightarrow 0 \quad (4.2)$$

contains a convergent subsequence in the norm of E .

Lemma 4.2 *Suppose that F satisfies (F₂), (F₈) and (F₃) as in (III). Then the functional Φ satisfies the (Ce) condition.*

Proof. Take a sequence pair $(u_n, v_n) \in E$ satisfying (4.2). It suffices to prove that $\|u_n\|_{W^{1,p}}$ and $\|v_n\|_{W^{1,q}}$ are bounded, as remarked in the proof of the previous lemma. It follows readily (4.2) that

$$\begin{aligned} C &\geq \left\langle \Phi'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n \right) \right\rangle - \Phi(u_n, v_n) \\ &= \int \left(\frac{1}{p}u_n F_u(u_n, v_n) + \frac{1}{q}v_n F_v(u_n, v_n) - F(u_n, v_n) \right). \end{aligned}$$

Then using (F₈) we obtain

$$\int |u_n|^\mu + \int |v_n|^\nu \leq \text{const}. \quad (4.3)$$

Next we use the following interpolation inequality, see [CM2]: let $0 < a < b < c$ and suppose that for some measurable function $u : \Omega \rightarrow \mathbb{R}$ we have that

$$\int |u|^a < \infty \quad \text{and} \quad \int |u|^c < \infty$$

then

$$\int |u|^b \leq \left(\int |u|^a \right)^{\frac{c-b}{c-a}} \cdot \left(\int |u|^c \right)^{\frac{b-a}{c-a}} \quad (4.4)$$

We use (4.4) for $0 < \mu < p < p^*$ and $0 < \nu < q < q^*$. So using (4.3) and (4.4) we get

$$\int |u_n|^p \leq C \left(\int |u_n|^{p^*} \right)^{\frac{p-\mu}{p^*-\mu}}, \quad \int |v_n|^q \leq C \left(\int |v_n|^{q^*} \right)^{\frac{q-\nu}{q^*-\nu}},$$

which implies by Sobolev imbedding

$$\int |u_n|^p \leq c \|u_n\|_{W^{1,p}}^{\tilde{p}}, \quad \int |v_n|^q \leq c \|v_n\|_{W^{1,q}}^{\tilde{q}}, \quad (4.5)$$

where

$$\tilde{p} = \frac{p - \mu}{p^* - \mu} p^*, \quad \tilde{q} = \frac{q - \nu}{q^* - \nu} q^*$$

Using (F_3) as in (III) we get

$$\Phi(u_n, v_n) \geq \frac{1}{p} \int |\nabla u_n|^p + \frac{1}{q} \int |\nabla v_n|^q - c \int |u_n|^p - c \int |v_n|^q$$

Which estimate by (4.5) leads to

$$\Phi(u_n, v_n) \geq \frac{1}{p} \|u_n\|_{W^{1,p}}^p + \frac{1}{q} \|v_n\|_{W^{1,q}}^q - c \|u_n\|_{W^{1,p}}^{\tilde{p}} - c \|v_n\|_{W^{1,q}}^{\tilde{q}}$$

Since $\Phi(u_n, v_n)$ is bounded and $\tilde{p} < p$ and $\tilde{q} < q$, it follows the boundedness of $\|u_n\|_{W^{1,p}}$ and $\|v_n\|_{W^{1,q}}$. $\square$

5 Proofs of Theorems completed

Theorem 1 *Condition (F_3) with r and s as in (I) implies that Φ is weakly lower semicontinuous and coercive in E . So, it assumes its infimum at a point $(u_0, v_0) \in E$. Condition (F_2) implies that this is a critical point Φ and consequently a weak solution of (1.2).*

Theorem 2 *As in Theorem 1, Φ assumes its infimum at a point $(u_0, v_0) \in E$. To prove that this is not $(0, 0)$ it is enough to show that there is $(u_1, v_1) \in E$ such that $\Phi(u_1, v_1) < 0$.*

Let φ be a first eigenfunction for the p -Laplacian

$$\begin{cases} -\Delta_p \varphi = \lambda_1(p) |\varphi|^{p-2} \varphi, & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and ψ for the q -Laplacian. We know that $\varphi, \psi \in C^{1,\alpha}$, see [dB], [T]. So we can take $\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty} \leq R$, where R is the constant in (F_5) . So

$$\begin{aligned} \Phi(t^{\frac{1}{p}} \varphi, t^{\frac{1}{q}} \psi) &\leq t \left\{ \frac{1}{p} \lambda_1(p) \int |\varphi|^p + \frac{1}{q} \lambda_1(q) \int |\psi|^q \right\} \\ &\quad - t^\theta \int K(x, \varphi, \psi), \end{aligned}$$

which is negative for $t > 0$ small.

Theorem 3 *It is easy to see that in this case Φ has the geometry of the Mountain Pass Theorem. Indeed, it follows from (F₃) and (F₇) that*

$$|F(x, u, v)| \leq c(|u|^{\bar{r}} + |v|^{\bar{s}} + |u|^r + |v|^s)$$

for all $x \in \bar{\Omega}$ and $(u, v) \in \mathbb{R}^2$, where $p < r, \bar{r} < p^*$ and $q < s, \bar{s} < q^*$. So by the Sobolev imbedding

$$\int F(x, u, v) \leq c(\|u\|_{W^{1,p}}^r + \|v\|_{W^{1,q}}^s + \|u\|_{W^{1,p}}^{\bar{r}} + \|v\|_{W^{1,q}}^{\bar{s}})$$

for all $(u, v) \in E$. So, we can estimate Φ by

$$\begin{aligned} \Phi(u, v) &\geq \frac{1}{p}\|u\|_{W^{1,p}}^p + \frac{1}{q}\|v\|_{W^{1,q}}^q \\ &\quad - c(\|u\|_{W^{1,p}}^r + \|v\|_{W^{1,q}}^s + \|u\|_{W^{1,p}}^{\bar{r}} + \|v\|_{W^{1,q}}^{\bar{s}}) \end{aligned}$$

which implies that there is an $\varepsilon > 0$ and $\rho > 0$ such that $\Phi(u, v) \geq \varepsilon$ for $\|u\|_{W^{1,p}} + \|v\|_{W^{1,q}} = \rho$. On the other hand, using (F₆) we have

$$\frac{d}{dt}\{F(x, t^{\theta_p}u, t^{\theta_q}v)\} \geq \frac{1}{t}F(x, t^{\theta_p}u, t^{\theta_q}v)$$

which implies that $F(x, t^{\theta_p}u, t^{\theta_q}v) \geq tK(x, u, v)$ for some function K . Since

$$\frac{1}{p} \int |\nabla(t^{\theta_p}u)|^p + \frac{1}{q} \int |\nabla(t^{\theta_q}v)|^q = t^{\theta_p p} c_1 + t^{\theta_q q} c_2$$

and from the hypothesis (F₆), $\theta_p p < 1$ and $\theta_q q < 1$, we conclude that for any fixed

$$(u, v) \neq (0, 0), \quad \Phi(t^{\theta_p}u, t^{\theta_q}v) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

Since Φ satisfies the (PS) condition, as proved in Lemma 4.1, we may apply the Mountain Pass Theorem and conclude.

Theorem 4 *It follows from (F₉) that given $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$F(x, u, v) \leq (a(x) + \varepsilon)G(u, v) + C_\varepsilon, \quad \text{all } x \in \bar{\Omega}, (u, v) \in R^2.$$

Hence

$$\Phi(u, v) \geq \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int (a + \varepsilon)G(u, v) - C_\varepsilon |\Omega|. \quad (5.1)$$

Now let $0 < \delta < 1$ be a real number to be chosen later, and write (5.1) as follows

$$\begin{aligned} \Phi(u, v) &\geq \delta \left\{ \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q \right\} \\ &\quad + (1 - \delta) \left\{ \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int \frac{a + \varepsilon}{1 - \delta} G(u, v) \right\} - c_\varepsilon |\Omega| \end{aligned}$$

The expression in the second bracket is estimated from below by

$$\lambda_1 \left(\frac{a + \varepsilon}{1 - \delta} \right) \left(\frac{1}{p} |u|^p + \frac{1}{q} \int |v|^q \right).$$

Now choose ε and δ such that $\lambda_1(\frac{a+\varepsilon}{1-\delta}) > 0$, which is possible in view of the continuity of $\lambda_1(a)$ with respect to a (Lemma 3.2), in the L^∞ -norm. So

$$\Phi(u, v) \geq \delta \left\{ \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q \right\} - C_\varepsilon |\Omega|$$

which shows that Φ is coercive. Since Φ is weakly lower semicontinuous, the result follows.

Theorem 5 *It suffices to show that Φ in this case has the geometry of the Mountain Pass Theorem. The compactness condition (Ce) has already been proved in Lemma 4.2. First we prove that $(0, 0)$ is a local minimum. Indeed, it follows from (F_{11}) and (F_3) that there is a constant $c > 0$ such that*

$$F(x, u, v) \leq c(x) \tilde{G}(u, v) + c(|u|^{\tilde{r}} + |v|^{\tilde{s}}) \quad (5.2)$$

for all $x \in \bar{\Omega}$, $(u, v) \in R^2$, and $\tilde{r} > p, \tilde{s} > q$. Next using (5.2) and a $\delta \in (0, 1)$ to be chosen later we have

$$\begin{aligned} \Phi(u, v) &\geq \delta \left(\frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q \right) \\ &\quad + (1 - \delta) \left\{ \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int \frac{c(x)}{1 - \delta} G(u, v) \right\} \\ &\quad - c \|u\|_{W^{1,p}}^{\tilde{r}} - c \|v\|_{W^{1,q}}^{\tilde{s}} \end{aligned}$$

Choosing δ such that the expression in the bracket is positive, as a consequence of (F_{11}) , we get

$$\Phi(u, v) \geq c_1 \|u\|_{W^{1,p}}^p + c_2 \|v\|_{W^{1,q}}^q - C \|u\|_{W^{1,p}}^{\tilde{r}} - c \|v\|_{W^{1,q}}^{\tilde{s}}$$

which shows that there are positive numbers ρ and ε such that $\Phi(u, v) > \varepsilon$ for $\|u\|_{W^{1,p}} + \|v\|_{W^{1,q}} = \rho$. Finally we use (F_{10}) to estimate

$$\Phi(u, v) \leq \frac{1}{p} \int |\nabla u|^p + \frac{1}{q} \int |\nabla v|^q - \int b(x) G(u, v) + C \quad (5.3)$$

Choosing $u = t^{1/p} u_0$ and $v = t^{1/q} v_0$, where (u_0, v_0) is the eigenfunction pair corresponding to $\lambda_1(b)$ and using Lemma 3.1 we get from

$$\begin{aligned} \Phi(t^{1/p} u_0, t^{1/q} v_0) &\leq \frac{t}{p} \int |\nabla u_0|^p + \frac{t}{q} \int |\nabla v_0|^q - t \int b G(u_0, v_0) + C \\ &= t \lambda_1(b) \left\{ \frac{1}{p} \int |u_0|^p + \frac{1}{q} \int |v_0|^q \right\} + C, \end{aligned}$$

which goes to $-\infty$ as $t \rightarrow +\infty$, in view of (F_{10}) .

6 Final comments

There are many open problems connected with the study of the functional Φ depending on other assumptions on the potential F .

- (i) Suppose that the growth of F with respect to u is like $|u|^{p^*}$ as $|u| \rightarrow \infty$. A similar question with respect to v . The scalar case has been studied by several people, Guedda-Veron [GV], Egnell [E]
- (ii) Problems with Ω replaced by $\mathbb{R}^N$. For example, in the scalar case see [0] and references there in.

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References

- [CM1] D.G. COSTA, C.A. MAGALHÃES, Variational elliptic problems which are non-quadratic at infinity, *Nonl. Anal. TMA* **23** (1994), 1401–1412.
- [FMT] P. FELMER, R.F. MANÁSEVICH, F. DE THÉLIN, Existence and uniqueness of positive solutions for certain quasilinear elliptic systems, *Comm. Part. Dif. Eq.* **17** (1992), 2013–2029.
- [V] J.L. VASQUEZ, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math and Optimization* **12** (1984), 191–202.
- [dT] F. DE THÉLIN, Première valeur propre d'un système elliptique non linéaire, *C.R. Acad. Sci. paris* **311** (1990), 603–606.
- [BFT] L. BOCCARDO, J. FLECKINGER, F. DE THÉLIN, Elliptic systems with various growths, preprint.
- [VT] J. VÉLIN, F. DE THÉLIN, Existence and non existence of nontrivial solutions for some nonlinear elliptic systems, *Rev. Mat. Univ. Compl. Madrid* **6** (1993), 153–194.
- [O] J.M.B. DO Ó, Nontrivial solutions for perturbations of p -Laplacian in $\mathbb{R}^N$ interacting with the first eigenvalue, preprint.
- [T] P. TOLKSDORF, Regularity for a more general class of quasilinear elliptic equations *J. Dif. Eq.* **51** (1984), 126–150.
- [CM2] D.G. COSTA, C.A. MAGALHÃES, A variational approach to non-cooperative elliptic systems, *Nonl. Anal. TMA* **25** (1995), 699–715.

- [GV] M. GUEDDA, L. VÉRON, Quasilinear elliptic equations involving critical Sobolev exponents, *Nonl. Anal. TMA* **13** (1989), 879–902.
- [E] H. EGNELL, Elliptic boundary value problems with singular coefficients and critical nonlinearities, *Indiana Univ. Math. J.* **38** (1989), 235–251.



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